Math. Proc. Camb. Phil. Soc. (2005), **138**, 107 © 2005 *Cambridge Philosophical Society* 107
DOI: 10.1017/S0305004104008011 *Printed in the United Kingdom*

Four-manifolds of large negative deficiency

BY CHARLES LIVINGSTON
Department of Mathematics, Indiana University, Bloomington, IN 47405, *U.S.A.*
e-mail: livingst@indiana.edu

(*Received* 27 *August* 2003; *revised* 10 *November* 2003)

Abstract

For every $N > 0$ there exists a group of deficiency less than $-N$ that arises as the fundamental group of a smooth homology 4-sphere and also as the fundamental group of the complement of a compact contractible submanifold of the 4-sphere. A group is the fundamental group of the complement of a contractible submanifold of the n-sphere, $n > 4$, if and only if it is the fundamental group of a homology n-sphere. There exist fundamental groups of homology n-spheres, $n > 4$, that cannot arise as the fundamental group of the complement of a contractible submanifold of the 4-sphere.

1. *Introduction*

The deficiency of a finite presentation of a group G is $g - r$, where g is the number of generators in the presentation and r is the number of relations. The deficiency of G, $\mathrm{def}(G)$, is the maximum value of this difference, taken over all presentations. The deficiency of a manifold, $\mathrm{def}(X)$, is defined to be $\mathrm{def}(\pi_1(X))$. Our goal is the proof of the following two theorems.

THEOREM 1·1. *For all $N > 0$ there exists a smooth 4-dimensional homology sphere X^4_N such that* $\mathrm{def}(X^4_N) < -N$.

THEOREM 1·2. *There exists a smooth, compact, contractible 4-manifold Z and for all $N > 0$ an embedding $f_N : Z \to S^4$ such that* $\mathrm{def}(S^4 - f_N(Z)) < -N$. *Furthermore, $Z \times I \cong B^5$ so, in particular, Z embeds in S^4 with contractible complement.*

Kervaire [**4**] proved that a group G is the fundamental group of a homology sphere of dimension greater than four if and only if G is finitely presented and $H_1(G) = 0 = H_2(G)$. Hausmann and Weinberger [**1**] demonstrated the existence of groups satisfying these conditions that cannot occur as the fundamental group of a homology 4-sphere. These groups have arbitrarily large negative deficiency and provided the first examples of the analog of Theorem 1·1 in high dimensions. Kervaire also proved that all perfect groups with deficiency 0 occur as the fundamental groups of homology 4-spheres and Hillman [**2**, **3**] recently constructed examples of homology 4-spheres of deficiency -1.

Lickorish [**7**] proved that any finitely presented perfect group G with $\mathrm{def}(G) = 0$ occurs as the fundamental group of the complement of a contractible submanifold of

S^4. In [**8**], Lickorish's construction was modified to build a contractible 4-manifold with an infinite number of embeddings into S^4, distinguished by the fundamental groups of the complement. The proof of Theorem 1·2 generalizes the construction of [**8**], giving the first complements of contractible 4-manifolds with negative deficiency.

Theorem 1·1 occurs naturally in the context of Kervaire's theorem. Theorem 1·2 can similarly be placed in the context of higher dimensional topology with the following theorem.

THEOREM 1·3. *For any $n \geqslant 5$ and group G, there exists a smooth, compact, contractible, n-dimensional submanifold $M_G \subset S^n$ with $\pi_1(S^n - M_G) = G$ if and only if G is finitely presented, perfect, and $H_2(G) = 0$. In this case $M_G \times I \cong B^{n+1}$, so, in particular, there exists an embedding $\phi: M_G \to S^n$ such that $S^n - \phi(M_G)$ is contractible.*

An extension of Hausmann and Weinberger's work yields the following theorem.

THEOREM 1·4. *There exist groups G that occur as the fundamental groups of the complements of compact contractible submanifolds in S^n for $n \geqslant 5$ but cannot occur as the fundamental groups of the complements of compact contractible manifolds in S^4.*

Notation and Conventions. We work in the smooth category. Homology and cohomology are taken with integer coefficients unless otherwise noted. The ring $\mathbf{Z}[t, t^{-1}]$ is denoted Λ.

2. *Deficiency*

Levine constructed knots in S^4 with complements of arbitrarily large negative deficiency [**6**]. His proof was based on the example of the 2-twist spin of the trefoil knot, for which he showed the complementary group has deficiency -1. The construction of 4-manifolds used here begins with a manifold W^4 which is then modified by replacing the tubular neighbourhood of a collection of embedded circles with complements of the 2-twist spin of the trefoil.

Levine's analysis of the deficiency of the groups depended on computations of the homology of the infinite cyclic cover of the corresponding spaces. This approach is not immediately available here, since the relevant spaces have trivial first homology and hence no infinite cyclic cover. The way around this difficulty is to work with manifolds that have finite covers which themselves do have infinite cyclic covers.

This section contains the necessary covering space theory, its application to the study of deficiency, and a review of Levine's results concerning the deficiency of Λ-modules. In the next section the applicable covering spaces of the 2-twist spin of the trefoil are studied. In Section 4 the general construction is described and the results of Sections 2 and 3 are applied to prove Theorems 1·1 and 1·2.

2·1. *Deficiency of finite index subgroups*

THEOREM 2·1. *If H is a subgroup of index n in a finitely presented group G, then* $\operatorname{def}(H) \geqslant n\operatorname{def}(G) - n + 1$.

Proof. A purely algebraic proof of this is based on the Reidemeister–Schreier rewriting process. Details are contained in [**9**], or see [**10**] for a covering space description. Briefly, G is the fundamental group of a 2-complex with one vertex, g 1-cells and r 2-cells. Hence, the n-fold cover with group H has a 1-skeleton built from

n 0-cells and ng 1-cells, so is homotopy equivalent to a 1-complex with one vertex and $ng - n + 1$ 1-cells. This has free fundamental group with $ng - n + 1$ generators. There are nr 2-cells, so H has a presentation with $ng - n + 1$ generators and nr relations, and hence deficiency $n(g - r) - n + 1$. The result follows.

2·2. *Deficiency and infinite cyclic covers*

Suppose that there is a surjective homomorphism ϕ from a finitely presented group G onto $\mathbf{Z}$ with kernel H. There is an action of $\mathbf{Z}$ on the abelianization of H, $H_1(H)$, making $H_1(H)$ into a Λ-module. In terms of covering space, if K_G is a space with fundamental group G, then $H_1(H)$ is the first homology of the (connected) infinite cyclic cover of K_G corresponding to H and the Λ-action is induced by the group of deck transformations.

Deficiency of Λ-modules. For an arbitrary finitely presented Λ-module M the deficiency is defined as for groups. The deficiency of a presentation is the difference of the number of generators and the number of relations. The deficiency of the module is the maximum value of this difference, taken over all presentations. It is bounded above by the rank of M; that is, by the dimension of the $\mathbf{Q}(t)$-vector space $M \otimes \mathbf{Q}(t)$.

As just described, if $\phi: G \to \mathbf{Z}$ is a surjective homomorphism, the abelianization of the kernel, H, is a Λ-module, $H_1(H)$. In this situation there is the following bound.

THEOREM 2·2. $\mathrm{def}(H_1(H)) \geqslant \mathrm{def}(G) - 1$.

Proof. If G has a presentation with g generators and r relations, then it follows from the Reidemeister–Schreier rewriting process, or the corresponding geometric construction, that $H_1(H)$ has a presentation with $g - 1$ generators and r relations as a Λ-module. The deficiency of this presentation is $g - 1 - r = (g - r) - 1$. The result follows.

2·3. *Iterated covers and deficiency*

THEOREM 2·3. *Suppose W is a connected manifold, $\tilde{W}$ is a connected n-fold cover of W, and $\phi: \pi_1(\tilde{W}) \to \mathbf{Z}$ is a surjection. If $\tilde{W}_\infty$ is the associated infinite cyclic cover of $\tilde{W}$, then*

$$\mathrm{def}(H_1(\tilde{W}_\infty)) \geqslant n(\mathrm{def}(\pi_1(W)) - n.$$

Proof. The result is an immediate corollary of Theorems 2·1 and 2·2.

2·4. *The deficiency of Λ-modules*

For a finitely presented Λ-module the following inequality was proved by Levine. The proof is summarized below, with relevant definitions embedded in the summary. Details can be found in [6].

THEOREM 2·4. *If M a finitely presented Λ-module of rank r_0 and deficiency d, then* $\mathrm{Ext}^2_\Lambda(M, \Lambda)$ *can be generated, as a Λ-module, by $r_0 - d$ elements.*

Proof. A finite presentation of M with g generators and r relations ($g - r = d$) yields an exact sequence

$$0 \longrightarrow F_2 \longrightarrow F_1 \xrightarrow{\phi_1} F_0 \longrightarrow M \longrightarrow 0,$$

with F_0 free of rank g and F_1 free of rank r. By definition, F_2 is the kernel of ϕ_1 and homological properties of Λ imply that F_2 is free. Tensoring with $\mathbf{Q}(t)$ yields an exact sequence of $\mathbf{Q}(t)$ vector spaces with dimensions, by definition, the ranks of each of the modules. Since the alternating sum of these dimensions is 0, $\operatorname{rank}(F_2) = r - g + r_0 = r_0 - d$.

By definition, $\operatorname{Ext}^2_\Lambda(M,\Lambda)$ is given as the cokernel of the induced map $\operatorname{Hom}_\Lambda(F_1,\Lambda) \to \operatorname{Hom}_\Lambda(F_2,\Lambda)$, and hence is a quotient of a free module of rank $r_0 - d$. It follows that it can be generated by $r_0 - d$ elements, as desired.

Example 2·5. The Λ modules Λ, $\Lambda/\langle 3\rangle$ and $\Lambda/\langle 3, t+1\rangle$ have deficiencies 1, 0, and -1, respectively.

The obvious presentations have these deficiencies, so it remains to see that these represent the maximum deficiencies. For Λ itself this is immediately seen by moving to the vector space setting via tensoring with $\mathbf{Q}(t)$. For $\Lambda/\langle 3\rangle$ the same argument applies, since $\Lambda/\langle 3\rangle \otimes \mathbf{Q}(t) = 0$.

For $\Lambda/\langle 3\ ,\ t+1\rangle$, notice that its rank, r_0, is 0. Hence, by applying Theorem 2·4 the question is reduced to determining whether $\operatorname{Ext}^2_\Lambda(\Lambda/\langle 3,t+1\rangle,\Lambda)$ can be generated by fewer than 1 element; that is, is $\operatorname{Ext}^2_\Lambda(\Lambda/\langle 3,t+1\rangle,\Lambda)$ trivial? The free resolution of $\Lambda/\langle 3,t+1\rangle$ is given by

$$0 \longrightarrow \Lambda \xrightarrow{\phi_2} \Lambda^2 \xrightarrow{\phi_1} \Lambda \longrightarrow \Lambda/\langle 3,t+1\rangle \longrightarrow 0,$$

where $\phi_1((1,0)) = 3$, $\phi_1((0,1)) = t+1$, and $\phi_2(1) = (t+1,-3)$. Applying Hom gives that $\operatorname{Ext}^2_\Lambda(\Lambda/\langle 3,t+1\rangle,\Lambda)$ is the cokernel of the map

$$\Lambda^2 \xrightarrow{\phi_2^*} \Lambda,$$

where $\phi_2^*((1,0) = t+1$ and $\phi_2^*((0,1) = -3$. Hence, $\operatorname{Ext}^2_\Lambda(\Lambda/\langle 3,t+1\rangle,\Lambda) \cong \Lambda/\langle t+1,-3\rangle$. As desired, this is nontrivial – as an abelian group it is isomorphic to $\mathbf{Z}_3$.

3. *Building blocks for high deficiency examples*

Let K denote the 2-twist spin of the trefoil knot in S^4. Let E denote the complement of an open tubular neighbourhood of K. This space is a fiber bundle over S^1 with fiber a punctured lens space, $L(3,1)_0$. The monodromy is an involution which acts nontrivially on $\mathbf{Z}_3 = H_1(L(3,1)_0)$ and the computation of the fundamental group follows readily:

THEOREM 3·1. $\pi_1(E) = \langle t, x \mid x^3 = 1, txt^{-1} = x^{-1}\rangle$.

There is a homomorphism of $\pi_1(E)$ to $\mathbf{Z}$ sending t to d and x to 0. This induces an infinite cyclic cover of E, denoted $\tilde{E}_d$. The usual example is the case of $d = 1$, for which $\tilde{E}_d \cong L(3,1)_0 \times \mathbf{R}$. If $d = 0$ the cover consists of an infinite family of copies E. If $d > 1$ the cover is disconnected, consisting of d copies of $\tilde{E}_1$.

Computing the first homology of this cover as a Λ-module follows from standard methods in the case of $d = 1$ and is a trivial calculation in the case of $d = 0$. For $d > 1$ the deck transformation T cyclically permutes the d copies of $\tilde{E}_1$, with T^d restricting to give the generating deck transformation on each of the copies of $\tilde{E}_1$. The next theorem is a consequence of these observations.

THEOREM 3·2. *For* $d > 0$, $H_1(\tilde{E}_d) \cong \Lambda/\langle 3, t^d + 1\rangle$. *For* $d = 0$, $H_1(\tilde{E}_d) \cong \Lambda$.

In order to apply a Mayer–Vietoris argument it is also necessary to understand the homology of the boundary of covers of E. However, the boundary of E is $S^1 \times S^2$, and the $d > 0$ infinite cyclic covers of the boundary have trivial first homology. In general:

THEOREM 3·3. *For $d > 0$, $H_1(\partial \tilde{E}_d) \cong 0$. For $d = 0$, $H_1(\partial \tilde{E}_d) \cong \Lambda$.*

Let W be a compact 4-manifold and $\phi: H_1(W) \to \mathbf{Z}$ be a surjection inducing an infinite cyclic cover, $\tilde{W}$. Suppose that α is an oriented simple closed curve in W and $\phi([\alpha]) = d$. Let W^* be the manifold constructed by removing a tubular neighbourhood of α and replacing it with E via any homeomorphism of the boundaries. Removing α does not change the first homology of W or of $\tilde{W}$. A Mayer–Vietoris argument gives the homology of the infinite cyclic cover $\tilde{W}^*$.

THEOREM 3·4. *If $d > 0$, $H_1(\tilde{W}^*) \cong H_1(\tilde{W}) \oplus \Lambda/\langle 3, t^d + 1\rangle$. If $d = 0$, $H_1(\tilde{W}^*) \cong H_1(\tilde{W})$.*

4. *Building the examples*

The starting point for our construction is a perfect group with specified properties. The simplest example is now described. Let G be the free product $D * D$ where D is the binary icosahedral group, the fundamental group of the Poincaré dodecahedral homology 3-sphere. The group D has order 120, is perfect, and has deficiency 0. (One presentation is $\langle x, y \mid x^2 = y^3 = (xy)^5\rangle$.) The homomorphism of $D * D$ to D given by isomorphisms on each factor has an index 120 kernel isomorphic to the free product of 119 copies of $\mathbf{Z}$. (To see this, let M be the dodecahedral space with the interior of a 3-ball removed and consider the restriction to $\partial(M \times I)$ of the (120-fold) universal cover of $M \times I$. It is clear that this cover corresponds to the kernel of the map on fundamental groups induced by the inclusion of $\partial(M \times I)$ into $M \times I$. Since the universal cover of M is a 120 times punctured 3-sphere, the boundary of the cover of $M \times I$ consists of two copies of S^3 each with 120 balls removed and with boundaries identified. The resulting space is $\#_{119} S^1 \times S^2$.)

For the proof of Theorem 1·1, let W be a homology 4-sphere with fundamental group G. It exists by [4]. For the proof of Theorem 1·2, let W be the complement of a contractible manifold Z embedded in S^4 with $\pi_1(W) = G$. By [7] such a Z exists, with the additional property that $Z \times I = B^5$.

Let $\tilde{W}$ be the associated 120-fold cover of W and let $\phi: \pi_1(\tilde{W}) \to \mathbf{Z}$ be an arbitrarily chosen surjective homomorphism. Let $\tilde{W}_\infty$ be the corresponding infinite cyclic cover.

Select an oriented simple closed curve x in $\tilde{W}$ such that $\phi([x]) = 1$. It can be assumed via general position that the projection of x, $\overline{x}$, in W is an embedding.

Replace each tubular neighbourhood of k parallel copies of $\overline{x}$ with E, the complement of the 2-twist spin of the trefoil. Call the resulting manifold Y_k. In the case that W is a homology 4-sphere, Y_k will also be a homology 4-sphere. In the case that W is the complement of a contractible manifold Z in S^4, since the operation of replacing the neighbourhood of a circle in S^4 with a knot complement gives S^4 back again if the correct glueing homeomorphism is used, it follows that Y_k embeds in S^4 with complement diffeomorphic to Z.

Let $\tilde{Y}_k$ denote the 120-fold cover of Y_k and let $\tilde{Y}_{k,\infty}$ denote the infinite cyclic cover of $\tilde{Y}_k$.

Suppose that $\pi_1(Y_k)$ has deficiency d. Then according to Theorem 2·3 the deficiency of $H_1(\tilde{Y}_{k,\infty})$ satisfies

$$\operatorname{def}(H_1(\tilde{Y}_{k,\infty})) \geqslant 120(d-1).$$

The space $\tilde{Y}_k$ is built from $\tilde{W}$ by removing parallel copies of neighbourhoods of x and replacing them with copies of E. (On each of these lifts ϕ takes value 1.) In addition, the other lifts of $\bar{x}$ are removed and replaced by E also. (Notice that since $\tilde{W}$ is a regular cover, the projection is a homeomorphism when restricted to each lift.) On each of these lifts the representation ϕ takes various values.

From this we see that as a Λ-module there is the following decomposition:

$$H_1(\tilde{Y}_{k,\infty}) = H_1(\tilde{W}_\infty) \oplus \left(\frac{\Lambda}{\langle 3, t+1\rangle}\right)^k \oplus \left(\oplus_i \frac{\Lambda}{\langle 3, t^{d_i}+1\rangle}\right)^k \oplus \left(\oplus_j \frac{\Lambda}{\langle 3\rangle}\right)^k.$$

(The range of the indices i and j have not been computed explicitly and may be empty. They will soon drop out of the calculations.)

Suppose now that $H_1(\tilde{Y}_{k,\infty})$ has deficiency D and that $H_1(\tilde{W}_\infty)$ is generated by N elements. Then, using the deficiency D presentation of $H_1(\tilde{Y}_{k,\infty})$ and adding N relations gives a deficiency $D-N$ presentation of the direct sum

$$M = \left(\frac{\Lambda}{\langle 3, t+1\rangle}\right)^k \oplus \left(\oplus \frac{\Lambda}{\langle 3, t^{d_i}+1\rangle}\right)^k \oplus \left(\oplus \frac{\Lambda}{\langle 3\rangle}\right)^k.$$

Hence, the deficiency of M satisfies $\operatorname{def}(M) \geqslant D-N$.

The Λ-module M is of rank 0 by definition; it is Λ-torsion. According to Theorem 2·4, in this case $\operatorname{Ext}^2_\Lambda(M,\Lambda)$ can be generated by $-\operatorname{def}(M)$ elements. In particular, its quotient,

$$\operatorname{Ext}^2_\Lambda\left(\left(\frac{\Lambda}{\langle 3, t+1\rangle}\right)^k\right) \cong \left(\frac{\Lambda}{\langle 3, t+1\rangle}\right)^k$$

could be generated by $-\operatorname{def}(M)$ elements. Hence, $-\operatorname{def}(M) \geqslant k$, or $\operatorname{def}(M) \leqslant -k$. We now have that $D-N \leqslant -k$, or $D \leqslant N-k$.

Applying Theorem 2·3 to this situation now gives that $\operatorname{def}(\pi_1(Y_k))$ satisfies

$$120\operatorname{def}(\pi_1(Y_k)) - 120 \leqslant \operatorname{def}(H_1(\tilde{Y}_{k,\infty})) \leqslant N-k.$$

We have

$$\operatorname{def}(\pi_1(Y_k)) \leqslant \frac{N+120-k}{120}.$$

Since this goes to $-\infty$ as k goes to infinity, the manifolds Y_k have the desired negative deficiencies and theorems 1·1 and 1·2 are proved.

5. *Higher dimensional complements*

In this section we prove Theorem 1·3.

THEOREM 1·3. *For any $n \geqslant 5$ and group G there exists a smooth, compact, contractible, n-dimensional submanifold $M_G \subset S^n$ with $\pi_1(S^n - M_G) = G$ if and only if G is finitely presented, perfect, and $H_2(G) = 0$. In this case $M_G \times I \cong B^{n+1}$, and in particular there exists an embedding $\phi\colon M_G \to S^n$ such that $S^n - \phi(M_G)$ is contractible.*

Proof. The second statement is automatic: if M_G is contractible then $M_G \times I$ is a smooth contractible manifold with boundary a homotopy sphere. By the h-cobordism theorem the boundary is diffeomorphic to S^n and contains M_G.

One direction of the first statement is immediate. Given the existence of such an M_G, the complement has finitely generated fundamental group and is a homology ball. Hence G is perfect. Since a $K(G,1)$ can be built from $S^n - M_G$ by adding cells of dimension 3 and higher, $H_2(G)$ is a quotient of $H_2(S^n - M_G) = 0$.

For the proof of the reverse implication, let N be an n-dimensional homology sphere with $\pi_1(N) = G$, the existence of which is given by [4]. Fix a handlebody structure on N and let N_k be the union of all handles of dimension k and less.

The homology group $H_2(N_2)$ is free, say on a generators, and hence there are b 3-handles, with $b \geqslant a$. The corresponding presentation matrix of $H_2(N_3)$ is given by a matrix A with b rows and a columns. Repeated row operations, corresponding to handle slides on the 3-handles, yields a matrix A' in upper triangular form; that is, $A'(i,j) = 0$ if $i > j$. Since $H_2(N_3) = 0$, the diagonal entries of this matrix must all be ± 1.

Let N_3' denote the union of N_2 with the first a of the 3-handles in this new handlebody decomposition. From the construction, $\pi_1(N_3') = G$ and N_3' is a homology ball.

In the argument that follows it is necessary that N_3' be stably parallelizable; in fact, N_3' has trivial tangent bundle, as seen as follows. Since N is orientable, the tangent bundle of N_1 is trivial. The obstruction to trivializing the tangent bundle over the 2-skeleton of N, or over N_2, is the second Stiefel-Whitney class, $w_2 \in H^2(N, \mathbf{Z}_2) = 0$. The obstruction to extending this trivialization over the 3-skeleton is in $H^3(N, \pi_2(SO(n)))$. But π_2 of any Lie group is trivial, so this obstruction vanishes also.

Since N_3' is a stably parallelizable homology ball of dimension 5 or higher, basic surgery theory [5] implies that surgery can be performed on N_3' to yield a contractible manifold M_G without changing the boundary. The boundary union $N_3' \cup_\partial M_G$ is a homotopy sphere, since the map $\pi_1(\partial N_3') \to \pi_1(N_3')$ is surjective. (To see this surjectivity, note that N_3' is built with handles of dimension 3 and less; hence there is a dual handlebody structure building N_3' from $\partial N_3' \times I$ using handles of dimension $n-3$ and higher. The surjectivity now follows from the assumption that $n - 3 \geqslant 2$.) Denote this homotopy sphere, $N_3' \cup_\partial M_G$, by Σ^n. The h–cobordism theorem implies that $\Sigma \# -\Sigma \cong S^n$.

Let $N_G = N_3' \# -\Sigma$. Then $N_G \cup_\partial M_G \cong S^n$. That is, M_G is a contractible manifold embedded in S^n with complement N_G having fundamental group G. This completes the proof.

6. *Groups that cannot occur in dimension* 4

The goal of this section is to demonstrate the existence of a finitely presented group G satisfying $H_1(G) = 0 = H_2(G)$ but such that G does not occur as the fundamental group of a compact homology 4-ball. The construction is a simple modification of the work of [1], to which the reader is referred for background.

Fix a prime p and for a group or space X let $\beta_i^p(X)$ denote the dimension of $H_i(X, \mathbf{Z}_p)$.

LEMMA 6·1. *If G is the fundamental group of compact 4-manifold M with nonempty connected boundary and $\chi(M) = 1$ (e.g. a rational homology 4-ball) and $H \subset G$ is index k, then*

$$2 + \beta_2^p(H) - 2\beta_1^p(H) \leqslant 2k.$$

Proof. Let $\tilde{M}$ be the k-fold cover of M corresponding to H. Then

$$\chi(\tilde{M}) = 1 - \beta_1^p(\tilde{M}) + \beta_2^p(\tilde{M}) - \beta_3^p(\tilde{M}) = k.$$

By duality, $\beta_3^p(\tilde{M}) = \beta_1^p(\tilde{M}, \partial\tilde{M})$. From the exact sequence,

$$H_1(\partial\tilde{M}, \mathbf{Z}_p) \longrightarrow H_1(\tilde{M}, \mathbf{Z}_p) \xrightarrow{\phi} H_1(\tilde{M}, \partial\tilde{M}, \mathbf{Z}_p) \longrightarrow H_0(\partial\tilde{M}, \mathbf{Z}_p) \longrightarrow H_0(\tilde{M}, \mathbf{Z}_p),$$

it follows that

$$\beta_1^p(\tilde{M}, \partial\tilde{M}) = (\beta_0^p(\partial\tilde{M}) - 1) + \mathrm{rank}(\mathrm{Im}(\phi)).$$

This in turn implies that

$$\beta_1^p(\tilde{M}, \partial\tilde{M})) \leqslant (k-1) + \beta_1^p(\tilde{M}).$$

Combining these shows that

$$\beta_3^p(\tilde{M}) \leqslant (k-1) + \beta_1^p(H).$$

Hence,

$$1 - \beta_1^p(H) + \beta_2^p(H) - ((k-1) + \beta_1^p(H)) \leqslant k.$$

Simplifying gives the desired inequality.

In [1] there is constructed, for each positive integer k, a group G satisfying the conditions of Theorem 1·3 that contains a subgroup A of finite index, where that index is independent of k. These groups have the property that $\beta_1^p(A)$ is bounded above by a linear function of k and $\beta_2^p(A)$ is bounded below by a quadratic function of k. The inequality of Lemma 6·1 cannot hold for all k, and hence these examples yield the proof of Theorem 1·4.

7. *Open problems*

There remains no characterization of the fundamental groups of homology 4-spheres. In addition, it is unknown whether such a characterization must depend on the category, smooth or topological.

Since in higher dimensions the set of fundamental groups of homology spheres is the same as the set of fundamental groups of complements of contractible manifolds, one can wonder if the same holds in dimension 4. We can state this as a conjecture.

CONJECTURE 7·1. *A group is the fundamental group of a homology 4-sphere if and only if it is the fundamental group of the complement of a contractible submanifold of the 4-sphere.*

Neither direction of the conjecture is known to be true.

REFERENCES

[1] J.-C. HAUSMANN and S. WEINBERGER. Caractéristiques d'Euler et groupes fondamentaux des variétés de dimension 4. *Comment. Math. Helv.* **60** (1985), 139–144.

[2] J. Hillman. Four-manifolds, geometries and knots. *Geom. Topol. Monogr.* **5** (2002).

[3] J. Hillman. An homology 4-sphere group with negative deficiency. *Enseign. Math.* (2) **48** (2002), 259–262.

[4] M. Kervaire. Smooth homology spheres and their fundamental groups. *Trans. Amer. Math. Soc.* **144** (1969), 67–72.

[5] M. Kervaire and J. Milnor. Groups of homotopy spheres. I. *Ann. of Math.* (2) **77** (1963), 504–537.

[6] J. Levine. Some results on higher dimensional knot groups, with an appendix by Claude Weber. In *Knot Theory* (Proc. Sem., Plans-sur-Bex, 1977). Lecture Notes in Math., 685 (Springer-Verlag, 1978), pp. 243–273.

[7] W. B. R. Lickorish. Knotted contractible 4-manifolds in the 4-sphere. *Pacific J. Math.* **208** (2003), 283–290.

[8] C. Livingston. Observations on Lickorish knotting of contractible 4-manifolds. *Pacific J. Math.* **209** (2003), 319–323.

[9] W. Magnus, A. Karrass and D. Solitar. *Combinatorial Group Theory. Presentations of Groups in Terms of Generators and Relations.* Second revised edition (Dover Publications Inc., 1976).

[10] J. Stillwell. *Classical Topology and Combinatorial Group Theory.* Second edition, Graduate Texts in Mathematics, **72** (Springer-Verlag, 1993).